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Random Walks with Infinite Spatial and Temporal Moments

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The continuous-time random walk of Montroll and Weiss has been modified by Scher and Lax to include a coupled spatial-temporal memory. We treat novel cases for the random walk and the corresponding generalized master equation when combinations of both spatial, and temporal moments of the memory are infinite. The asymptotic properties of the probability distribution for being at any lattice site as a function of time and its variance are calculated. The resulting behavior includes localized, diffusive, wavelike, and Levy's stable laws for the appropriate scaled variable. We show that an infinite mean waiting time can lead to long time diffusive behavior, while a finite mean waiting time is not sufficient to ensure the same.

KEY WORDS: Random walk; coupled memory; infinite moments; stable distributions.

1. INTRODUCTION

Random walks lie at the heart of much analysis in stochastic processes. In addition a large variety of physical phenomena involving transport, configurational statistics, fluctuations in the state of a system, etc. can be mapped onto a random walk problem. The appropriate random walk may, however, be semi-Markovian, non-Markovian, involve internal states, a coupled memory, a high number of dimensions, a defective lattice, difficult boundary conditions, or other complications. The random walk need not take place in a real positional space and the transitions (events) need not represent actual physical jumps. Recent review articles cover many of these

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cases,[1-3] and Montroll combines several of the above conditions in a random walk approach to the kinetic Ising model.[4] It is the purpose of the present paper to study the behavior of a coupled spatial–temporal memory random walk especially in the novel case when both spatial and temporal moments are infinite.

An important random variable representing the random walk is the sum S_N, where

$$S_N = X_1 + \cdots + X_N \tag{1}$$

and the X_i are identically distributed random variables each with mean μ and variance σ^2. If the variance is finite then the central limit theorem can be invoked to obtain the Gaussian probability density, say in one dimension,

$$\begin{aligned} f(x) &= \lim_{N\to\infty} \text{Prob.}\left[x < S_N/\sqrt{N} < x + dx \right] \\ &= (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2) \end{aligned} \tag{2}$$

We have set $\mu = 0$ and will do so for the rest of the analysis because one can define a new variable $Y = X - \mu$ which has a zero mean. Also we will only discuss one-dimensional cases here.

One may further introduce a probability density $\psi(t)$ governing the time between the events X_i, and study the sum $S_{N(t)}$, where

$$S_{N(t)} = X_1 + \cdots + X_{N(t)} \tag{3}$$

The random variable $N(t)$ represents the number of events which have occurred in the time interval $[0, t]$. If $\psi(t)$ has a finite first moment $\bar{t}$ and $\sigma^2 < \infty$, then again the central limit theorem can be used to show that[10]

$$\begin{aligned} f(x,t) &= \lim_{t\to\infty} \text{Prob.}\left[x < S_{N(t)}/\sqrt{t} < x + dx \right] \\ &= (4\pi D t)^{-1/2} \exp(-x^2/4Dt) \end{aligned} \tag{4}$$

where $D = \sigma^2/2\bar{t}$.

At this point it may appear that Gaussian behavior is inevitable when summing identically distributed random variables. However, the Gaussian limit can be avoided when one of the three cases below occurs:

1. The second moment σ^2 of the probability density $p(X)$ of the random variable X is infinite.
2. The probability density $\psi(t)$ has an infinite first moment.
3. A combination of infinite moments of both $p(X)$ and $\psi(t)$ occur.

The first case was studied by P. Levy,[5] who showed that $f(x) \equiv \lim_{N\to\infty} \text{Prob.}\ (x < S_N/N^{1/\beta} < x + dx)$ has the simple form in Fourier space $(x \leftrightarrow k)$

$$f(k) = \exp(-b|k|^\beta), \qquad 0 < \beta \leqslant 2 \tag{5}$$

with b a real positive constant when $\mu = 0$. Two well-known examples are the Cauchy distribution

$$f(x) = \frac{1}{\pi b} \frac{1}{(x/b)^2 + 1}, \qquad \beta = 1 \tag{6}$$

and the Smirnov distribution $\beta = \frac{1}{2}$,

$$f(x) = \frac{1}{\sqrt{\pi}} \frac{2}{b^2} \left(\frac{b^2}{2x} \right)^{3/2} \exp\left(\frac{-b^2}{2x} \right), \qquad x > 0 \tag{7}$$

The case $\beta = 2$ occurs when $\sigma^2 < \infty$ and it is the Gaussian distribution. For other β, $f(x)$ is only known in terms of infinite series.[6] Properties of random walks on lattices with infinite second moments of the single jump probability distribution have been given by Gillis and Weiss.[7] The clustering nature of the random walk paths for stable distributions has recently been discussed.[18]

The second case with $\psi(t) \sim t^{-1-\alpha}$, $0 < \alpha < 1$, so $\bar{t} = \int_0^\infty t\psi(t)\,dt = \infty$ has been discussed by Feller[6] and further analyzed by Montroll and Scher[8,9] and by Shlesinger[10] to model charge transfer in amorphous media, such as xerographic films. Such a waiting time density will lead to a zero dc conductivity in the decoupled Scher–Lax[11] model of conduction and the name "localization condition" is applied. Essentially, the central limit theorem breaks down, even though the X's have finite moments, because there is a high probability that none or just a few events have occurred by any time t.

Both cases 1 and 2 have been reviewed by Montroll and West[1] and by Weiss and Rubin.[3] It is the purpose of this manuscript to further the study of random walk behavior by considering case 3, where both spatial and temporal moments are infinite.

2. RANDOM WALK FRAMEWORK

We chose to analyze our stochastic process as a random walk on an infinite periodic lattice with all lattice sites being equivalent and we will follow in the spirit of Montroll and Weiss,[13] and Scher and Lax.[11] The former authors[13] introduced the continuous-time random walk while the latter authors[11] were the first to use a coupled memory random walk in their theory of impurity conduction. We will assume the walker always begins at the origin $l = 0$ at time $t = 0$.

First, we introduce a coupled memory $\Psi(l, t)$, which is a probability density governing single transitions, i.e., $\Psi(l, t) =$ probability density that a transition of displacement l occurs at a time t after the previous transition.

The probability $\psi(t)\,dt$ that a transition occurs in the time interval $(t, t + dt)$ is given by

$$\psi(t)\,dt = \sum_l \Psi(l, t)\,dt \tag{8a}$$

Furthermore, we require the normalization

$$\int_0^\infty \psi(t)\,dt = 1 \tag{8b}$$

The probability density $R(l, t)$ for reaching site l *exactly* at time t, which involves summing over many different paths, can be calculated in terms of $\Psi(l, t)$, which governs single transitions, by using a recursion relation and assuming translational invariance,

$$R(l, t) = \sum_{l' \neq 0} \int_0^t \Psi(l', \tau) R(l - l', t - \tau)\,d\tau + \delta(t)\delta_{l,0} \tag{9}$$

The probability $P(l, t)$ for being at site l at time t is related to $R(l, t)$ by

$$P(l, t) = \int_0^t R(l, t - \tau)\Phi(\tau)\,d\tau \tag{10}$$

where

$$\Phi(\tau) = 1 - \sum_l \int_0^\tau \Psi(l, t)\,dt \tag{11}$$

is the probability that no transition occurs in the time interval $(0, \tau)$. Equation (10) takes into account that the walker reached site l at an earlier time $t - \tau$, and then no transition takes place in the remaining time τ. Fourier transforming over the lattice $(l \to k)$ and Laplace transforming over time $(t \to s)$ one finds[11] with

$$P(k, s) \equiv \sum_l \int_0^\infty P(l, t) e^{ikl} e^{-st}\,dt$$

that

$$P(k, t) = \mathfrak{L}^{-1}\left[\frac{1}{1 - \Psi(k, s)} \frac{1 - \psi(s)}{s} \right] \tag{12}$$

where $\mathfrak{L}^{-1}$ is the inverse Laplace transform and $\psi(s) = \Psi(k = 0, s)$. The second moment of $P(l,t)$ is given by[11]

$$\begin{aligned}\langle l^2(t)\rangle &= \sum_l l^2 P(l,t) \\ &= -\left.\frac{\partial^2}{\partial k^2} P(k,t)\right|_{k=0} \\ &= -\mathcal{L}^{-1}\left[\frac{\partial^2\Psi(k,s)/\partial k^2}{[1-\Psi(k,s)]^2}\,\frac{1-\psi(s)}{s}\right]_{k=0}\end{aligned} \tag{13}$$

In the following sections we will study the possible behaviors of $P(l,t)$ and its variance.

3. DECOUPLED MEMORY

3.1. Finite Moments

Let us begin by considering that the memory $\Psi(l,t)$ can be decoupled into a spatial and a temporal part,

$$\psi(l,t) = \psi(t)p(l) \tag{14}$$

as discussed in the Introduction. If all the moments of $\psi(t)$ and $p(l)$ exist then their Laplace ($t \leftrightarrow s$) and Fourier ($l \leftrightarrow k$) transforms have the following expansions:

$$\psi(s) = 1 - \bar{t}s + \tfrac{1}{2}\,\overline{t^2}\,s^2 + \cdots \tag{15a}$$

$$p(k) = 1 - \tfrac{1}{2}\,\overline{l^2}\,k^2 + \tfrac{1}{24}\,\overline{l^4}\,k^4 + \cdots \tag{15b}$$

As a first example let us choose our lattice with unit bond length,

$$\psi(t) = \lambda\exp(-\lambda t) \tag{16a}$$

and

$$p(l) = \tfrac{1}{2}(\delta_{l,1} + \delta_{l,-1}) \tag{16b}$$

Using Eq. (15) in Eqs. (12) and (13) directly leads to[14]

$$P(l,t) = \exp(-\lambda t)I_l(\lambda t) \tag{17a}$$

and for large times

$$\langle l^2(t)\rangle = 2Dt \tag{17b}$$

where I_l is the imaginary Bessel function of lth order and $D = \lambda\overline{l^2}/2 = \lambda/2$ since $\sum_l l^2 p(l) = 1$. In the long time limit Eq. (17a) will approach the Gaussian distribution (Brownian motion) of Eq. (4). This behavior will be found as long as $\overline{l^2}$ and $\bar{t}$ are finite, as can be seen substituting Eq. (15) into Eq. (12)

$$\lim_{t\to\infty}\lim_{k\to 0} P(k,t) \sim \mathfrak{L}^{-1}\left[\frac{\bar{t}}{s\bar{t}+\overline{l^2}k^2/2}\right]$$
$$= \exp\left[-\left(\overline{l^2}/2\bar{t}\right)k^2 t\right]$$

which upon inverse Fourier transforming yields

$$\lim_{t\to\infty}\lim_{l\to\infty} P(l,t) \sim (4\pi D t)^{-1/2}\exp(-l^2/4Dt) \tag{18}$$

3.2. Infinite Spatial Moments

If $\bar{t}$ is finite, but

$$\lim_{l\to\infty} p(l) \sim \frac{1}{|l|^{1+\beta}}, \qquad 0<\beta<2$$

then $\overline{l^2}$ will be infinite, and

$$\lim_{k\to 0} p(k) \sim 1 - A|k|^{\beta} \tag{19}$$

with A a constant,[7] in contrast to the moment expansion, Eq. (15b). This will lead to

$$\lim_{t\to\infty}\lim_{k\to 0} P(k,t) \sim \exp\left(-A|k|^{\beta}t/\bar{t}\right) \tag{20}$$

which is the Fourier transform of a Levy distribution Eq. (5). Not much information is gained by studying the second moment, since one arrives at using Eq. (13),

$$\left\langle l^2(t)\right\rangle = \infty \tag{21}$$

because

$$\partial^2 p(k)/\partial k^2\big|_{k=0} = \infty$$

In Section 4, where the memory function $\Psi(l,t)$ is not decoupled, we will see how to retain the time dependence of $\langle l^2(t)\rangle$ even when the *analog* of $\overline{l^2}$ is infinite.

3.3. Infinite Temporal Moments

If now $\overline{l^2}$ is finite, but $\psi(t) \sim t^{-1-\alpha}$, $0<\alpha<1$, so $\bar{t}$ is infinite then, with c a constant,

$$\lim_{s\to 0}\psi(s) \sim 1 - cs^{\alpha} \tag{22}$$

as opposed to the moment expansion in Eq. (15a). Analyzing Eq. (12) for small k and s we find

$$P(k,s) \sim \left(cs^{\alpha} + \overline{l^2}k^2/2\right)^{-1} s^{\alpha-1} \tag{23}$$

which in real space has the following form for large l:

$$P(l,s) = \frac{s^{\alpha-1}\left\{\exp\left[-(s^{\alpha}/D)^{1/2}|l|\right]\right\}}{2D^{1/2}s^{\alpha/2}}$$

where $D = \overline{l^2}/2c$. Returning to a distribution function we have

$$\begin{aligned} P(l \geqslant L,s) &= \sum_{l=L}^{\infty} P(l,s) \\ &\sim (1/2s)\exp\left[-L(s^{\alpha}/D)^{1/2}\right] \end{aligned} \tag{24}$$

After inverse Laplace transforming, Eq. (24) has the form of a stable distribution S, of order $\alpha/2$, of a scaled variable, i.e.,

$$P(l \geqslant L,t) = \tfrac{1}{2}S_{\alpha/2}\left[t/\left(L^2/D\right)^{1/\alpha}\right] \tag{25}$$

as has been shown by Tunaley.[12] Of course, for $\alpha = 1$, this is an equivalent way of writing the Gaussian distribution in Eq. (18). Using this momentless $\psi(t)$ in Eq. (13) gives[10]

$$\lim_{t\to\infty} \left\langle l^2(t)\right\rangle = Dt^{\alpha}/\Gamma(1+\alpha) \tag{26}$$

Thus in the decoupled scheme when either $\overline{l^2}$ or $\bar{t}$ are infinite, non-Gaussian behavior results.

4. COUPLED MEMORY

For a specific example of how a coupled memory can arise consider a particle which is scattered at random times. Its mean free path is then a random variable. The longer the time between scattering events the longer will be the mean free path. Translated into random walk terminology, with jumps corresponding to collisions, the distance a particle jumps will depend on the time since the previous jump. This necessitates a coupled spatial temporal memory.

With a decoupled memory and $\overline{l^2}$ infinite we obtained $\langle l^2(t)\rangle = \infty$ for all $t > 0$. With a coupled memory it is possible to examine the temporal behavior of $\langle l^2(t)\rangle$ as it grows in time. Scher and Lax[11] were the first to stress that $\Psi(l,t)$ should not be decoupled and recently Klafter and Silbey[15] have shown in a specific and exact case how a coupled memory can lead to a crossover between coherent and incoherent exciton transport.

Weiss[16] has analyzed coupled memory random walks for cases leading to Gaussian distributions and shown how the coupling effects the mean and variance, especially when a bias is present. We choose to write the coupled memory as

$$\Psi(l,t) = \psi(t)p(l\,|\,t) \tag{27a}$$

where

$$\sum_l p(l\,|\,t) = 1 \tag{27b}$$

and

$$\int_0^\infty \psi(t)\,dt = 1 \tag{27c}$$

so

$$\int_0^\infty \sum_l \Psi(l,t)\,dt = 1$$

The probability density $\psi(t)$ governs the time between events and $p(l\,|\,t)$ is a conditional probability that the jump (transition) goes a distance l given that it takes place a time t after the preceding jump. Of course, if $p(l\,|\,t)$ does not depend on time we have a decoupled momory. Note in Eq. (27b) we need only a sum over l for the proper normalization.

First, following Weiss[16] let us choose $\psi(t)$ to have finite moments,

$$\psi(t) = \lambda\exp(-\lambda t)$$

and choose $p(l\,|\,t)$ to behave at long times and distances as

$$p(l\,|\,t) = \frac{1}{(4\pi Dt)^{1/2}}\exp(-l^2/4Dt) \tag{28}$$

where

$$\sigma^2(t) \equiv \sum_l l^2 p(l\,|\,t) = 2Dt$$

Then in Eq. (13)

$$-\left.\frac{\partial^2\Psi(k,s)}{\partial k^2}\right|_{k=0} = -\mathfrak{L}\left[\psi(t)\frac{\partial^2 p(k\,|\,t)}{\partial k^2}\right]_{k=0}$$

$$= L\left[\lambda\sigma^2(t)e^{-\lambda t}\right] = \frac{2\lambda D}{(s+\lambda)^2} \sim \frac{2D}{\lambda} \tag{29}$$

The denominator in Eq. (12) is equal to $1/(s^2\bar{t})$, so we find asymptotically

$$\langle l^2(t)\rangle = \left(\frac{D}{\lambda\bar{t}}\right)t \tag{30}$$

Note that asymptotic in space and time

$$\Psi(k,s) \sim \Psi(k=0,s) + \tfrac{1}{2}\left[\partial^2\Psi(k=0,s)/\partial k^2\right]k^2 + \cdots \tag{31}$$

$$\sim 1 - s\bar{t} + (D/\lambda)k^2 \tag{32}$$

Using Eqs. (30) and (31) in Eq. (12) will yield Gaussian behavior for $P(l,t)$, since $\Psi(k,s)$ effectively factors into $\psi(s)p(k)$ for small s and k. *We want to find the conditions under which* $\Psi(k,s)$ *will not decouple.* First, note that $p(l\,|\,t)$ was chosen to be a probability distribution with a single jump length variance $\sigma^2(t)$ growing in time as $2Dt$. We could have alternatively chosen

$$p(l\,|\,t) = J_l^2(\lambda t) \tag{33a}$$

so

$$\sigma^2(t) = \sum_l l^2 p(l\,|\,t) = -\left.\frac{\partial^2 p(k\,|\,t)}{\partial k^2}\right|_{k=0} \sim \frac{\lambda^2 t^2}{2} \tag{33b}$$

Since for $p(l\,|\,t)$ we only require normalization with respect to l we could choose asymptotically

$$p(l\,|\,t) = \frac{1}{\left[2\pi d(t)\right]^{1/2}} \exp\left[-\frac{l^2}{2d(t)}\right] \tag{34}$$

yielding

$$\sigma^2(t) = d(t) \tag{35}$$

with $d(t)$ any positive function of t. So in general in Eq. (13)

$$\left.\frac{-\partial^2\Psi(k,s)}{\partial k^2}\right|_{k=0} = \mathfrak{L}\left[\psi(t)d(t)\right] \tag{36}$$

If at large times

$$\psi(t)d(t) \sim ct^n/\Gamma(n+1), \qquad n > -1 \tag{37a}$$

then

$$\mathfrak{L}\left[\psi(t)d(t)\right] \sim cs^{-(n+1)} \qquad \text{for small } s \tag{37b}$$

where c is a constant. Equations (37) can be generalized by allowing c to be a slowly varying function. In any event Eqs. (37) imply that $\Psi(k,s)$ cannot be decoupled, i.e.,

$$\lim_{s\to 0}\lim_{k\to 0}\Psi(k,s) \sim \psi(s) - \frac{c}{2}\,\frac{k^2}{s^{n+1}} \tag{38}$$

The possible behaviors of $\psi(s)$ were discussed in Section 3, while Eq. (37) gives classes of behavior for the second term in the $\Psi(k,s)$ expansion Eq. (31), when $\Psi(k,s)$ cannot be decoupled. If $\psi(t)d(t)$ decays more rapidly algebraically then in Eq. (37a) its Laplace transform will asymptotically approach a constant (as can be seen by expanding the exponential in the Laplace transform) and $\Psi(k,s)$ will be decoupled into a $\psi(s)p(k)$, as in Eq. (29).

Consider now the specific case

$$\psi(t) \sim t^{-1-\alpha}, \qquad 0 < \alpha < 1 \tag{39a}$$

and

$$d(t) \sim t^{m}, \qquad m > 0 \tag{39b}$$

The mean time $\bar{t}$ between jumps is infinite, thus yielding

$$d(\bar{t}) = \sum_{l} l^2 p(l \mid \bar{t}) = \infty$$

In this sense, the mean time between jumps is infinite as well as the average mean squared displacement per jump.

If $m > \alpha$, then, asymptotically

$$\Psi(k,s) \sim 1 - as^{\alpha} - \frac{b}{2} \frac{k^2}{s^{m-\alpha}} \tag{40}$$

with a and b constant. Substituting this $\Psi(k,s)$ and $\psi(s) = \Psi(k=0,s)$ into Eq. (13) yields

$$\langle l^2(t) \rangle = \frac{b}{a} \frac{t^m}{\Gamma(m+1)} \tag{41}$$

Note for $m = 1$ the effects of an infinite mean waiting time and an infinite second moment jump displacement off-set each other and yield the Brownian motion result for the mean squared displacement. We have thus demonstrated that depending on the coupled memory, $\bar{t} < \infty$ is not necessary to obtain long time diffusive behavior. To see this more clearly we examine the asymptotic behavior of $P(k,s)$ in Eq. (12) to find

$$P(k,s) \sim \frac{1}{as^{\alpha} + bk^2/2s^{m-\alpha}} \left(\frac{as^{\alpha}}{s} \right) = \frac{s^{m-1}}{s^m + bk^2/2a} \tag{42}$$

which for $m = 1$ is the Gaussian case of Eq. (18). If $2 > m > \alpha$, we arrive back at the stable distribution of order $m/2$ as in Eq. (25) for the scaled variable $t/l^{2/\alpha}$. If $m < \alpha$ we have $\Psi(k,s)$ decoupling as in Section 3.

For $m = 2$ we have asymptotically with, $v^2 = b/2a$,

$$P(k,s) = \frac{s}{s^2 + v^2k^2} = \frac{1}{2}\left[\frac{1}{s + ivk} + \frac{1}{s - ivk} \right]$$

As discussed by Kadanoff and Swift,[17] complex poles should lead to wave behavior. Inverse Laplace transforming yields

$$P(k,t) = \cos(vkt)$$

and finally inverse Fourier transforming yields (in the continuum limit)

$$P(l,t) = \tfrac{1}{2}\big(\delta(l - vt) + \delta(l + vt)\big) \tag{43}$$

which represents coherent wave motion.

As an example with $\bar{t}$ finite, but $\overline{t^2}$ infinite we choose

$$\psi(t) \sim 1/t^{2+\gamma}, \qquad 0 < \gamma < 1 \tag{44a}$$

and

$$d(t) \sim t^m \tag{44b}$$

Then asymptotically, if $m > 1 + \gamma$,

$$\Psi(k,s) \sim 1 - s\bar{t} - \frac{bk^2}{2s^{m-1-\gamma}} \tag{45}$$

and

$$P(k,s) \sim \frac{s^{m-1-\gamma}}{s^{m-\gamma} + (b/2\bar{t})k^2} \tag{46}$$

For $m - \gamma \leqslant 1$ we again obtain Brownian motion. If $m - \gamma = 2$ wave motion is obtained. This complements the result of Klafter and Silbey[15] that $\bar{t} < \infty$ is not sufficient for a Gaussian behavior. In Ref. 15 it has been shown that $\bar{t} < \infty$ and a coupled memory may give rise to wave (coherent) motion.

As a last example consider a *coupled memory where both $\overline{l^2}$ and $\bar{t}$ are infinite*. The quantity $\Psi(k, s = 0)$ which appears in the study of the probability of a random walker returning to its origin, easily exhibits the nature of a random walk with both spatial and temporal infinite moments:

$$\begin{aligned}
\Psi(k, s = 0) &= \int_0^\infty p_\beta(k \mid t)\psi_\alpha(t)\, dt \\
&= \int_0^\infty e^{-A|k|^\beta t}\psi_\alpha(t)\, dt \\
&= \psi_\alpha(s = A|k|^\beta) \sim 1 - \text{const}|k|^{\alpha\beta}
\end{aligned} \tag{47}$$

where we have chosen $p(l \mid t)$ to be a β-stable process, Eq. (5), and $\psi_\alpha(t) \sim t^{-1-\alpha}$, $0 < \alpha < 1$. Thus, we have arrived at a stable process of order $\alpha\beta$. This is an example of Bochner's[6] subordination technique.

5. GENERALIZED MASTER EQUATION FRAMEWORK

In the above discussion we have assumed:

1. Random walks on periodic lattices
2. $\Psi(l,t)=\psi(t)p(l\,|\,t)$.

These assumptions may now be relaxed.

It has been shown that for a lattice randomly occupied by guest sites the average probability $\langle P(l,t)\rangle$ obeys a generalized master equation[19]

$$\frac{d}{dt}\langle P(l,t)\rangle=\sum_{l'}\int_0^t d\tau\, M(l-l',t-\tau)\langle P(l',\tau)\rangle \tag{48}$$

where $M(l-l',\tau)$ is the self-energy, and $P(l,t)$ has been averaged over all configurations of the lattice guest sites (for details see Ref. 19). The self-energy is translationally invariant and so Fourier–Laplace-transforming Eq. (48) we find

$$\langle P(k,s)\rangle=\left[s-M(k,s)\right]^{-1} \tag{49}$$

Conservation of probability is equivalent to $M(k=0,s)=0$. Relations between the generalized master equation, Eqs. (48) and (49), and the coupled memory random walk used by Scher and Lax,[11] Eq. (12), have been established[15,19,20] in a manner similar to the decoupled case[21]:

$$\Psi(k,s)=\psi(s)+\frac{1-\psi(s)}{s}M(k,s) \tag{50}$$

as can also be seen by equating the right-hand sides of Eqs. (12) and (49).

Expanding $M(k,s)$ around $k=0$ we obtain

$$M(k,s)\sim k^2F(s) \tag{51a}$$

with

$$F(s)=\frac{1}{2}\,\frac{\partial^2 M(k,s)}{\partial k^2} \tag{51b}$$

From Eqs. (50), (51), and (36), we see that the choice for $\Psi(l,t)$, Eqs. (27a) and (34), is related to the above derivation in the following way:

$$\frac{1-\psi(s)}{s}F(s)=\mathfrak{L}\left[\psi(t)d(t)\right] \tag{52}$$

The variance of $\langle P(l,t)\rangle$ is given by

$$\langle l^2(t)\rangle=-\mathfrak{L}^{-1}\left[\frac{2F(s)}{s^2}\right] \tag{53}$$

We now classify $\langle l^2(t)\rangle$ according to $F(s)$. For long times we find

$$\langle l^2(t)\rangle = \begin{cases} t^n & \text{if } F(s) \sim -s^{1-n}, \quad 1 > n \geqslant 0 \\ t & \text{if } F(s) \sim \text{const} \\ t^2 & \text{if } F(s) \sim -s^{-1} \end{cases} \tag{54}$$

Using Eq. (50) we may derive the corresponding $\Psi(k,s)$ for each $\psi(s)$ of Section 3. We recover the localization, diffusion, and wave (coherent) behavior which were derived in Section 4, and here we did not have to specify a particular form for $\Psi(l,t)$ as in Eqs. (27a) and (34).

5. CONCLUSIONS

We have considered continuous-time random walks on lattices governed by a memory [the waiting time distribution $\Psi(l,t)$] and shown under what conditions the memory will remain coupled in space and time. A larger class of behavior is possible when the memory asymptotically remains coupled in space and time with interesting cases occurring when the spatial and temporal moments of the memory are infinite. We showed that with a coupled memory a finite mean waiting time is neither necessary nor sufficient to lead to long time diffusive behavior. We stressed the realization $\Psi(l,t) = \psi(t)p(l \mid t)$, where $\psi(t)$ governed the time between events and $p(l \mid t)$ governed the jump displacement given the knowledge of when the last jump occurred. However, in our generalized master equation approach we worked directly with the memory kernel $M(l,t)$ and did not need to resort to a particular realization.

A simple example where a coupled memory appears naturally concerns a particle being scattered at random times in random directions with a constant velocity between collisions. The longer the time between scattering events the longer will be the mean free path. If the mean time between collisions is infinite then wave motion will result. This is in contrast to a decoupled memory random walk where an infinite mean time between jumps leads to a localized behavior.

It is possible that the random process will pass through several interesting stages before reaching its final asymptotic form. For example, Klafter and Silbey[15] have shown that in the case of a one-dimensional exciton interacting with local stochastic fluctuations $\Psi(k,s)$ can be derived with $F(s) \sim -(s+A)^{-1}$, A being a constant. A crossover from wave behavior $[F(s) \sim -s^{-1}, |s| \gg A]$ to diffusive behavior $[F(s) \sim -A^{-1}, |s| \ll A]$ occurs. The crossover behavior of other random processes derived from physical considerations will be investigated, where the crossover is intrinsic or induced by varying external parameters. Different classes of $d(t)$ and the statistics they generate will also be investigated.

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